Enrollment No: ______________________ Exam Seat No:______________________

C.U.SHAH UNIVERSITY

Winter Examination-2015

Subject Name: Advanced Functional Analysis

Subject Code: 5SC03AFE1 **Branch: M.Sc. (Mathematics)**

Semester: 3 Date: 10/12/2015 Time:2:30 To 5:30 Marks: 70

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions.** **(07)**

a. Let $X = K \times K \times \ldots = \prod_{i=1}^{\infty} K_i$, where $K_i = K, \forall i$. **(01)**
Is $\|x\|_p = \left(\sum_{j=1}^{\infty} |x(j)|^p\right)^{\frac{1}{p}}, x \in X$ norm on X?

b. Let $r, p \in [1, \infty]$ with $r < p$. What is relation between l^p and l^r. **(01)**

c. Let Y be a subspace of a norm linear space X. Is it true that Y and $\bar{Y}$ are norm linear spaces with the norm on ? **(01)**

d. State Holder's inequality. **(02)**

e. State F. Riesz's Lemma. **(02)**

Q-2 **Attempt all questions** **(14)**

a) Let $a_j, b_j \in K, (j = 1,2, \ldots n)$ and $1 \le p \le \infty$. Prove that **(07)**

$$\left(\sum_{j=1}^{n} |a_j + b_j|^p\right)^{1/p} \le \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{1/p}$$

b) Define norm linear space. **(07)**

Let $l^p = \left\{x = (x(j))_{j=1}^{\infty} \mid \|x\|_p = \left(\sum_{j=1}^{\infty} |x(j)|^p\right)^{\frac{1}{p}} < \infty\right\}$ prove hat $(l^p, \|\cdot\|_p)$ is a norm linear space.

OR

Q-2 **Attempt all questions** **(14)**

a) Let $\left(X_j, \|\cdot\|_j\right) (j = 1,2, \ldots, m)$ be norm linear space and **(07)**

$X = \prod_{j=1}^{\infty} X_j = X_1 \times X_2 \times \ldots \times X_m$. If $\|x\|_p = \left(\sum_{j=1}^{m} \|x(j)\|_j^p\right)^{\frac{1}{p}}$ then prove the following

(i) $\|x\|_p$ is a norm on X.

(ii) If a sequence $\{x_n\}$ in X converges to $x \in X$ with respect to $\|x\|_p$ if and only if $x_n(j) \to x(j)$ in X_j for each $j = 1,2, \ldots, m$.

b) Let X and Y be normed linear spaces. If X is finite dimensional then prove that every linear map from X to Y is continuous. **(07)**

Q-3 **Attempt all questions** **(14)**

a) Let F be a linear map from a norm linear space X to a norm linear space Y then prove the following. **(07)**

(i) Prove that F is homeomorphism if and only if there exist $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|, \ \forall\ x \in X$.

(ii) If F is on to and linear homeomorphism then prove that X is complete if and only if Y is complete.

b) State and prove Hahn Banach separation theorem. **(07)**

OR

Q-3 **Attempt all questions** **(14)**

a) Let X be an norm linear space then prove that the following are equivalent **(07)**

(i)$X$is a Banach space.

(ii) Every absolutely summable series in $X$is summable.

b) Let X be an norm linear space, Y be a subspace of X and $g \in Y'$. Prove that there exists $f \in X'$ such that $f/Y = g$and $\|f\| = \|g\|$. **(07)**

SECTION – II

Q-4 **Attempt the Following questions.** **(07)**

a. Define: Strictly convex space. **(02)**

b. Define: Banach Space with help of an example. **(02)**

c. State Open mapping theorem. **(02)**

d. For any normed linear space X, X' is always complete. Determine whether statement is true or false? **(01)**

Q-5 **Attempt all questions** **(14)**

a) Let X be a complex norm linear space. **(07)**

(i) If $u: X \to R$ be a linear functional Define $f: X \to \mathbb{C}$ by $f(x) = u(x) - i\ u(ix)$. Prove that f is a complex linear functional.

(ii) If f is linear functional and define $u(x) = Re\ (f(x))$, prove that u is a real linear functional and $f(x) = u(x) - i\ u(ix)$.

b) Let X, Y be norm linear spaces. **(07)**

(i) Let $E \subset X$. Then prove that E is bounded if and only if $f(E)$ is bounded in $K, \forall\ f \in X'$.

(ii) Let $F: X \to Y$ be linear. Prove that F is continuous if and only if $gof: X \to K$ is continuous for every $g \in Y'$.

OR

Q-5 **Attempt all questions** **(14)**

a) Let X be a Banach space, Y be a normed linear spaceand $\mathcal{A}$ be subset of $B(X, Y)$ such that $\{Ax : A \in \mathcal{A}\}$ is bounded for each $x \in X$. Prove that $\mathcal{A}$ is a bounded subset of $B(X, Y)$. **(07)**

b) Let X, Y be metric spaces and $f: X \to Y$ be a function. **(07)**

(i) Prove that f is a closed map if and only if f has a closed graph.

(ii) Prove that f is continuous then f has closed graph. Is converse true? Justify your answer.

Q-6 **Attempt all questions** **(14)**

a) State and prove closed graph theorem. **(07)**

b) Let $1 \le p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Define $F: l^q \to (l^p)'$ by $F(y) = f_y$ where $y \in l^q, f_y \in (l^p)'$ is given by $f_y(x) = \sum_{j=1}^{\infty} x(j)y(j)$. Prove that F is an onto linear isometry. **(07)**

OR

Q-6 **Attempt all Questions**

a) Let X be a norm linear space and $A \in B(X)$ be of finite rank. Then prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$. **(07)**

b) (i) Let X be a norm linear space and $A \in BL(X)$. Prove that A is invertible if and only if A is bounded below and surjective. **(07)**

(ii) Let X be a complex. Prove that A is invertible if and only if A is bounded below and $R(A)$ is dense in X.